Recap: Matrix inversion.

Alg: To invert Matrix $M \in Mat_{n\times n}(\mathbb{R})$:

$$[M \mid I_n] \xrightarrow{\text{RREF}} [I_n \mid M^{-1}]$$

NB: if the RREF of $[M \mid I_n]$ does not have form $[I_n \mid M^{-1}]$, then it is NOT possible to invert...

Prop: Let A be an $m\times k$ matrix and B be a $k\times n$ matrix. Then $\boxed{L_B \circ L_A = L_{BA}}$.

Point: The matrix transformations have composition determined by the corresponding matrix product.

Pf: Skipped in lecture, feel free to request a video :).

Cor: Matrix multiplication is associative.

pf (Cor): Suppose A, B, C are matrices w/ "correct sizes for multiplication." We have:

$$L_{A(BC)} = L_A \circ L_{BC} = L_A \circ (L_B \circ L_C)$$

$$= (L_A \circ L_B) \circ L_C = L_{AB} \circ L_C = L_{(AB)C}$$

Hence $A(BC) = (AB)C$. $\boxtimes$

NB: If A is $m\times n$ and B is $k\times l$, then

$$L_A : \mathbb{R}^n \to \mathbb{R}^m \quad \text{and} \quad L_B : \mathbb{R}^l \to \mathbb{R}^k$$

If $m \neq \ell$, then $\mathbb{R}^n \xrightarrow{L_A} \mathbb{R}^m$

$\qquad\qquad\qquad\qquad\qquad \not\parallel$

$\qquad\qquad\qquad\qquad\qquad \mathbb{R}^\ell \xrightarrow{L_B} \mathbb{R}^k$,

So $L_B \circ L_A$ does not exist, same with

$B \cdot A$ is undefined...

Also recall, a map $L : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism when L^{-1} exists.

<u>Prop</u>: A map $L : \mathbb{R}^n \to \mathbb{R}^n$ is an automorphism when the matrix $[L]$ determining L is invertible.

I.E. when $[L^{-1}] = [L]^{-1}$ exists.

in particular, $[L] \cdot [L]^{-1} = I_n = [L]^{-1} \cdot [L]$.

$$I_n = [id_{\mathbb{R}^n}]$$

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* It turns out the invertible matrices have a decomposition as a product of "Elementary matrices".

<u>Def^n</u>: Let $n \geq 1$. An <u>elementary $n \times n$ matrix</u> is a matrix obtained from I_n via a single row operation.

① $M_i(c) \leftarrow$ multiply row i by $c \neq 0$.

② $P_{i,j} \leftarrow$ Swap row i and row j.

③ $A_{i,j}(c) \leftarrow$ add c times row i to row j (replace row j)

Ex: For $n=3$.

$$M_1(5) = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad M_3(\pi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pi \end{bmatrix}$$

$$P_{1,2} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad P_{1,3} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A_{1,3}(5) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} \qquad A_{3,1}(5) = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Prop: Matrix M is invertible if and only if M can be expressed as a product of elementary matrices.

Lem: The elementary matrices simulate row operations.
i.e. If E is an elementary matrix, then EM is the matrix obtained by applying the operation E represents to M.

Ex: $P_{1,3} \cdot M = \begin{bmatrix} \text{matrix obtained by swapping rows} \\ \text{1 and 3 in } M \end{bmatrix}$.

NB: Lemma proof is very simple... what remains follows from an induction on the number of row operations performed on the invertible matrix to reach the identity.

Ex: Express the (invertible!) matrix

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix}$$ as a product of elementary matrices.

Idea: Apply row reductions and record the inverse reduction...

Sol: $\begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix} \overset{\ell_1 \leftrightarrow \ell_2}{\rightsquigarrow} P_{2,1} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 2 \end{bmatrix}}$

$\rightsquigarrow P_{2,1}\left(A_{1,2}(1) \cdot \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix}\right)$ check $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix}$

$\rightsquigarrow P_{2,1} A_{1,2}(1) A_{1,3}(1) \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$

$\rightsquigarrow P_{2,1} A_{1,2}(1) A_{1,3}(1) \left(M_3(2) \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & \frac{1}{2} \end{bmatrix}\right)$

$\rightsquigarrow P_{2,1} A_{1,2}(1) A_{1,3}(1) M_3(2) A_{2,3}(1) \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$

$\rightsquigarrow P_{2,1} A_{1,2}(1) A_{1,3}(1) M_3(2) A_{2,3}(1) M_3(\tfrac{1}{2}) \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

$\rightsquigarrow P_{2,1} A_{1,2}(1) A_{1,3}(1) M_3(2) A_{2,3}(1) M_3(\tfrac{1}{2}) A_{3,2}(-1) \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$\rightsquigarrow P_{2,1} A_{1,2}(1) A_{1,3}(1) M_3(2) A_{2,3}(1) M_3(\tfrac{1}{2}) A_{3,2}(-1) A_{3,1}(-1) \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

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$\rightsquigarrow P_{2,1} A_{1,2}(1) A_{1,3}(1) M_3(2) A_{2,3}(1) M_3(\tfrac{1}{2}) A_{3,2}(-1) A_{3,1}(-1) A_{2,1}(1) M_3(-1) I_3$

Remarks: ① The factorization above is **NOT** the most "efficient" one...

② All the "$\rightsquigarrow$" should be replaced w/ "$=$"... what we computed were honest matrix equalities :) .

Prop: Let A be an $m \times n$ matrix. Then A can be expressed as $A = E_n E_{n-1} \cdots E_2 E_1 \, \mathrm{RREF}(A)$ for $E_1, E_2, \cdots, E_n$ elementary $m \times m$ matrices.

NB: This is essentially the same as saying A can be reduced to $\mathrm{RREF}(A)$ via elementary row operations.

Ex: Compute the inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ provided it exists.

Sol:

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|cc} ac & bc & c & 0 \\ ac & ad & 0 & a \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|cc} ac & bc & c & 0 \\ 0 & ad-bc & -c & a \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|cc} ac & bc & c & 0 \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$\frac{c(ad-bc)+bc^2}{ad-bc} = \frac{adc - \cancel{bc^2} + \cancel{bc^2}}{ad-bc}$$

$$\rightsquigarrow \left[\begin{array}{cc|cc} ac & 0 & c + \frac{bc^2}{ad-bc} & -\frac{abc}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$\therefore \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

<u>So</u>: If $ad-bc \neq 0$, then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

<u>Point</u>: Quantity $ad-bc$ is important: it determines whether or not $L_{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}$ is an automorphism.